Quantum-mechanical research on nonlinear resonance and the problem of quantum chaos

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(Received 26 November 2003; revised manuscript received 16 April 2004; published 30 August 2004)

The quantum-mechanical investigation of nonlinear resonance in terms of approximation to moderate nonlinearity is reduced to the investigation of eigenfunctions and eigenvalues of the Mathieu-Schrodinger equation. The eigenstates of the Mathieu-Schrodinger equation are nondegenerate in a certain area of pumping amplitude values in the neighborhood of the classical separatrix. Outside this area, the system finds itself in a degenerate state for both small and large pumping amplitude values. Degenerate energy terms arise as a result of merging and branching of pairs of nondegenerate energy terms. Equations are obtained for finding the merging points of energy terms. These equations are solved by numerical methods. The main objective of this paper is to establish a quantum analog of the classical stochastic layer formed in the separatrix area. With this end in view, we consider a nonstationary quantum-mechanical problem of perturbation of the state of the Mathieu-Schrodinger equation. It is shown that in passing through the branching point the system may pass from the pure state to the mixed one. At multiple passages through branching points there develops the irreversible process of "creeping" of the system to quantum states. In that case, the observed population of a certain number of levels can be considered, in our opinion, to be a quantum analog of the stochastic layer. The number of populated levels is defined by a perturbation amplitude.

DOI: 10.1103/PhysRevE.70.026219 PACS number(s): 05.45.Mt

I. INTRODUCTION: PROBLEM STATEMENT

Usually, quantum chaos means phenomena arising in a quantum system at those parameter values for which a classical analog of the system has properties of dynamic stochasticity. The Hamiltonian of such a system is written in the form

$$\hat{H} = H_o + \lambda \hat{V}, \tag{1}$$

where $\hat{H}_o$ is the integrable part of the Hamiltonian. As experimental studies show [1,2], the main property characteristic of quantum chaos is the repulsion of quasienergy terms $E_n(\lambda)$ at those values of the parameter λ for which, in the case of classical consideration, there arises dynamic stochasticity. Difficulties encountered in studying quantum chaos make it necessary to pass from the quantum-mechanical description to the quantum-statistical one. The general theory of the quantum-statistical description of systems in a state of chaos is presented in the monograph by Haake [3]. In this paper we will consider the case in which the integrable part $\hat{H}_o$ is reduced to the universal Hamiltonian, and the total Hamiltonian is of the form

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_o(l) + \hat{V}(t), \tag{2}$$

where $\hat{V}(t)$ is the periodical interaction. In the considered Hamiltonian (2) all difficulties of the quantum-mechanical description are connected not with the total Hamiltonian as in the case of model (1), but with its integrable part $\hat{\mathrm{H}}_o$. A detailed analysis of eigenstates of the universal Hamiltonian shows that the spectrum $E_n^{(0)}(l)$ has a complicated and specific dependence on the parameter l since it contains degenerate and nondegenerate domains separated by numerous branching points of the energy spectrum. Passages through the branching points due to the periodic interaction of $V(t)$ can lead to the mixed state that can be considered as a manifestation of quantum chaos. The example of the universal Hamiltonian considered in this paper is interesting because, like all integrable systems, it allows us to define precisely all quantum characteristics. However, these characteristics are rather complicated to serve as the base of quantum chaos.

At the classical reviewing the motion generated by the universal Hamiltonian on the phase space consists of two types of topologically distinguished curves divided by the separatrix [4]. As a consequence of the perturbation of trajectories near the separatrix by the periodic perturbation there arises a stochastic "layer"—the area of stochastic motion near the separatrix. At quantum reviewing Schrodinger equation for the universal Hamiltonian is represented in the form of the equation of Mathieu, solutions of which (wave functions) are the periodic functions of Mathieu.

The Mathieu equation for the quantum-mechanical description of the atom, having a nonlinear energy spectrum and subjected to the action of a resonance periodic field, was obtained by Zaslavsky and Berman [5].

Many physical problems can be reduced to the solution of the Mathieu equation. For example, in view of research of the phenomenon of parametrical resonance, the analysis of the Mathieu equations leads to the origin of zones of unstable motion [6]. In the study of motion of the electron in the periodic potential field, the analysis of the Schrodinger equation, which is reduced to the equation of the Mathieu, explains the presence of forbidden energy zones in semiconductors [7]. It is possible to cite other similar examples, however, the Mathieu-Schrodinger equation for the universal Hamiltonian differs from them, because in this case the motion is finite and hence the energy spectrum without fail is discrete. Depending on the values of the parameters of the problem, the energy spectrum can become degenerate. In the present work the condition of degeneration of the energy

spectrum and, the reason for their emerging will be investigated.

The main purpose of this paper is to describe a quantum analog of the classical stochastic layer. For this, in the last section we consider a nonstationary problem of eigenstate perturbation for the Mathieu-Schrodinger equation. It is shown that when passing through a branching point the system may transit from the pure state to the mixed one. The mixed state formed as a result of multiple passages through branching points is, in our opinion, a quantum analog of the stochastic layer.

II. UNIVERSAL HAMILTONIAN

Let us present the atom as a nonlinear oscillator under the action of the variable monochromatic field. Then the Hamiltonian of the system atom + field is of the form

$$H(x,p,t) = H_o(x,p) + H_{NL}(x) + \varepsilon V(x,t), \tag{3}$$

where

$$H_o = 1/2\left(\frac{p^2}{m} + \omega_o^2 m x^2\right), \quad H_{NL} = \beta x^3 + \gamma x^4 + \ldots, \tag{4}$$

$$V(x,t) = V_o x \cos \Omega t, \quad V_o = -\frac{e}{m} f_o, \quad \varepsilon \ll 1, \tag{5}$$

where e,m are the charge and mass of the electron, x,p are the coordinate and impulse of the electron, ω_o is the eigenfrequency, γ and β are the coefficients at the nonlinear terms, f_o and Ω are the amplitude and frequency of a variable field. Interaction $V(x,t)$ of the electron with the variable field we shall consider as small perturbation.

Having made passage to the variables of action angle (I,θ) with the help of transformations $x = (2I/m\omega_o)^{1/2}\cos\theta, p = -(2Im\omega_o)^{1/2}\sin\theta$, assuming that resonance condition $\dot{\theta} \approx \Omega$ is fulfilled, and averaging the equation with respect to fast phase θ from the Hamiltonian (3) we get

$$H(I,\varphi,t) = H_o^{NL} + \varepsilon V(I)\cos\varphi, \tag{6}$$

$$H_o^{NL} = \omega_o I + \overline{H_{NL}}, \quad \overline{H_{NL}} = \frac{3}{2}\left(\frac{I}{m\omega_o}\right)^2 \gamma, \tag{7}$$

$$V(I) = V_0 (I/2m\omega_0)^{1/2}. \tag{8}$$

Here we have introduced the slow phase $\varphi = \theta - \Omega t$, which in time $\approx 2\pi/\Omega$ varies insignificantly. The resonance condition is fulfilled for the particular value of the action I_o, the value of which can be found from the same resonance condition

$$\dot{\varphi} \approx \omega(I_o) = 0, \tag{9}$$

where

$$\omega(I) = \left(\frac{\partial H_0^{NL}}{\partial I}\right) - \Omega.$$

Introducing the dimensionless parameter of nonlinearity [4]

$$\varepsilon_1 = \left|\frac{d\omega}{dI}\right| \frac{I}{\omega} \tag{10}$$

and expanding Hamiltonian (7) in the series with respect to small deviations of the action $\Delta I = I - I_o$, in the approximation of the moderate nonlinearity

$$\varepsilon \ll \varepsilon_1 \ll 1/\varepsilon, \quad \Delta I/I_o \ll 1, \tag{11}$$

we obtain Hamiltonian in the form

$$H = \frac{\omega'}{2}(\Delta I)^2 + V\cos\varphi, \tag{12}$$

where

$$V = \varepsilon V(I_o), \quad \omega' = \left.\frac{d\omega}{dI}\right|_{I=I_o}. \tag{13}$$

Hamiltonian (12), called universal, as it is easy to note, is similar to the Hamiltonian of the pendulum with "mass" $1/\omega'$ and with "impulse" in the "gravity" field with acceleration of gravity $g \sim V$.

The value $1/\omega'$ is in fact the measure of system inertness: the higher $1/\omega'$, the more difficult it is to take the system out of resonance by means of pumping and, vice versa, the lower $1/\omega'$, the easier it is to do so. If in Eq. (12) ΔI is substituted by the appropriate operator $\Delta I \to -i\hbar\partial/\partial\varphi$, one can obtain the universal Hamiltonian in the quantum form

$$H = -\frac{\hbar^2\omega'}{2}\frac{\partial^2}{\partial\varphi^2} + V\cos\varphi. \tag{14}$$

Hamiltonian (14) describes the joint motion of the atom + field system. Here ω' is the only parameter characterizing the atom, while the pumping amplitude V is the parameter characterizing the field. $\hbar^2\omega'$ is a minimal energy value connected with nonlinearity and called quantum energy of nonlinearity.

III. SYMMETRIES OF THE MATHIEU-SCHRODINGER EQUATION

Having written the stationary Schrodinger equation,

$$\hat{H}\psi_n = E_n\psi_n, \tag{15}$$

for the Hamiltonian (14), we get

$$\frac{\partial^2\psi_n}{\partial\varphi^2} + [E_n - V(l,\varphi)]\psi_n = 0, \tag{16}$$

$$V(l,\varphi) = l\cos 2\varphi, \tag{17}$$

where the dimensionless quantities are introduced,

$$E_n \to \frac{8E_n}{\hbar^2\omega'}, \quad l \to \frac{8V}{\hbar^2\omega'}, \tag{18}$$

and the replacement $\varphi \to 2\varphi$ is done.

The Mathieu-Schrodinger equation was studied by Zaslavsky and Berman in the quasiclassical approximation [5]. In this work we shall investigate the Mathieu-

TABLE I. The relations of a symmetry for the Mathieu functions.

$G(\varphi)$	$G(-\varphi)$	$G(\pi-\varphi)$	$G(\pi+\varphi)$
$ce_{2m}(\varphi)$	$ce_{2m}(\varphi)$	$ce_{2m}(\varphi)$	$ce_{2m}(\varphi)$
$ce_{2m+1}(\varphi)$	$ce_{2m+1}(\varphi)$	$-ce_{2m+1}(\varphi)$	$-ce_{2m+1}(\varphi)$
$se_{2m+1}(\varphi)$	$-se_{2m+1}(\varphi)$	$se_{2m+1}(\varphi)$	$-se_{2m+1}(\varphi)$
$se_{2m+2}(\varphi)$	$-se_{2m+2}(\varphi)$	$-se_{2m+2}(\varphi)$	$se_{2m+2}(\varphi)$

Schrodinger equation in an essentially quantum area.

As is known [8], periodic solutions of Eq. (16) are given by the Mathieu functions

$$ce_{2m}(l,\varphi), \quad ce_{2m+1}(l,\varphi), \quad se_{2m+1}(l,\varphi), \quad se_{2m+2}(l,\varphi), \tag{19}$$

which satisfy the normalization condition

$$\frac{1}{\pi}\int_{o}^{2\pi} \psi_n^2(l,\varphi)d\varphi = 1, \tag{20}$$

where $\psi_n(l,\varphi)$ means Mathieu functions (19). To eigenfunctions (19) there correspond the eigenvalues (Mathieu characteristics)

$$a_{2m}(l), \quad a_{2m+1}(l), \quad b_{2m+1}(l), \quad b_{2m+2}(l), \tag{21}$$

which also depend on the parameter l.

The properties of symmetry of the Mathieu function can be presented [8] in the form of Table I. By immediate check it is easy to be convinced that four elements of transformation

$$G(\varphi \to -\varphi) = a, \quad G(\varphi \to \pi - \varphi) = b,$$

$$G(\varphi \to \pi + \varphi) = c, \quad G(\varphi \to \varphi) = e \tag{22}$$

form a group. For this purpose it is enough to test the realization of the following relations:

$$a^2 = b^2 = c^2 = e,$$

$$ab = c, \quad ac = b, \quad bc = a. \tag{23}$$

Group G contains three elements a,b,c of the second order and unity element e. The group G is isomorphic to the well-known group of Klein [9,10]. This group is known in group theory by the applications to quantum mechanics. All the elements of the group commute. This assertion can be easily checked taking into account group operations (23). So, the symmetry group of the Mathieu function G is the Abelian group and has only one-dimensional indecomposable representations.

Therefore Mathieu functions (19) corresponding to the symmetry group G describe nondegenerate states.

The group of transformations G is not a simple group since it contains subgroups. When combined with the unit element, each of three elements a,b,c forms a subgroup of second order:

$$G \to \begin{cases} \nearrow & G_+ : e,b, \\ & G_- : e,c, \\ \searrow & G_o : e,a. \end{cases} \tag{24}$$

Moreover, as can be easily verified, these subgroups are invariant subgroups.

As is known from the theory of groups, the existence of subgroups indicates the existence of degenerate states of the system with a higher symmetry than the symmetry defined by the basic group G. Since in our case these subgroups are invariant subgroups of second order, we may expect the occurrence of degenerate states of second order. Thus, by virtue of general arguments based on the symmetry properties of the Mathieu-Schrodinger equation, we conclude that the system has both degenerate and nondegenerate states.

IV. DEGENERATE STATES OF THE MATHIEU-SCHRODINGER EQUATION

In the theory of Mathieu functions, the graphs of the eigenvalues $a_n(l)$ and $b_n(l)$ as functions of l are plotted by numerical methods (11) and (12). As seen from these graphs, curves $a_n(l)$ and $b_n(l)$ merge for small l, while curves $a_n(l)$ and $b_{n+1}(l)$ merge for large l. It is obvious that the merged segments of the Mathieu characteristics correspond to the degenerate states whose existence has been mentioned above. In this section, we will define the wave functions of degenerate states and obtain equations for finding the merging (or branching) points of the Mathieu characteristics. Below, the presence of branching points will play an essential in role in explaining the transition from the pure state to the mixed one during the quantum investigation of the dynamics near the classical separatrix.

In what follows, we will use the plane with coordinates l,E. In the classical consideration, the motion of a mathematical pendulum in a neighborhood of the separatrix occurs when the initial kinetic energy of the pendulum is close to the maximal potential one. It is obvious that, on the plane (l,E), to this condition there corresponds the straight line $l=E$. Therefore we can say that, on the plane (l,E), to nondegenerate states there corresponds a certain domain lying on both sides of the line $l=E$. It is in this very domain of the change of l that the system is characterized by symmetry group G.

A. Degeneration of states at small l: Area on the left of the separatrix line

In the limit $l \to 0$ the equation of the Mathieu-Schrodinger equation (16) takes the form

$$\frac{d^2\psi_n}{d\varphi^2} + E_n\psi_n = 0. \tag{25}$$

The orthonormalized system of solutions of Eq. (25) consists of even and odd solutions

$$\psi_g = \cos n\varphi, \quad \psi_u = \sin n\varphi. \tag{26}$$

They both correspond to the same energy value $E_n = n^2$, i.e., for $l=0$ there occurs a doublefold degeneration of levels.

Note that functions (26) correspond to the well-known limiting ($l \to 0$) forms of Mathieu functions [11]:

$$ce_n(\varphi) \to \cos n\varphi, \quad se_n(\varphi) \to \sin n\varphi. \tag{27}$$

This means that at the diminution of l the coming together of the energy terms with the identical n takes place and for $l=0$ they are merged together. It is necessary to find out that this confluence happens at the point $l=0$ or at $l=l_-^{(n)} \neq 0$. In this section, below we will be concerned with finding a lower point of the merging of terms $l_-^{(n)}$.

At first let us find out what the eigenfunctions of the degenerated states corresponding to the level $E_n=n^2$ look like. Equation (25) is the Schrodinger equation for free rotation in the phase plane φ. The continuous Abelian group of two-dimensional rotations $O^+(2)$ [9] corresponds to this motion.

Since the Abelian group may have only one-dimensional irreducible representations, the two-dimensional representation constructed in the base of real-valued functions (26) will be reducible. Hence functions (26) cannot be eigenfunctions of a degenerate state. To surmount this problem we shall recollect that the eigenfunctions for the degenerate condition can be also complex.

As is known [10], symmetry relative to the time sign change in the Schrodinger equation accounts for the fact that the complex-conjugate wave functions correspond to one and the same energy eigenvalue. Therefore two complex-conjugate representations $\psi_n(\varphi)$ and $\psi_n^*(\varphi)$ should be regarded as a representation of doubled dimension [9,10]. Usually, for the base of the indecomposable representation of the group $O^+(2)$ complex functions are taken [9],

$$\psi_n(\varphi) = e^{-in\varphi}. \tag{28}$$

So, in the degenerate area in view of conditions of normalization [see Eq. (20)], for eigenfunctions complex conjugate functions should be taken

$$\psi_n(\varphi) = \frac{\sqrt{2}}{2} e^{-in\varphi}, \quad \psi_n^*(\varphi) = \frac{\sqrt{2}}{2} e^{in\varphi}. \tag{29}$$

Let us remark that group $O^+(2)$ is isomorphic to subgroup G_- (24). The element of symmetry $c=G(\varphi \to \pi+\varphi)$ of subgroup G_- provides recurrence of the phase variation after each period and consequently the symmetry G_- characterizes the condition of motion similar to the classical rotary motion.

However, to use only the argument of symmetry is not sufficient for finding the coordinates of the branching point $l_-^{(n)}$. Below to find these points we use the secular perturbation theory.

So, at $l=0$ we have doubly degenerate states with the wave functions (29). Let us find out, whether the perturbation

$$V(l,\varphi) = l \cos 2\varphi, \quad l \ll 1 \tag{30}$$

can remove the existing degeneration.

As is known, first order terms of the perturbation theory for the energy eigenvalues and the exact functions of zero approximation for double degenerate levels look like [10]

$$E_{o\pm}^{(1)} = \frac{1}{2}[(V_{11}+V_{22}) \pm \sqrt{(V_{11}+V_{22})^2 + 4|V_{12}|^2}], \tag{31}$$

$$\psi_n^{\pm} = \psi^{(o)} = C_1\psi_1^o + C_2\psi_2^o, \tag{32}$$

$$C_1^{(0)} = \left\{ \frac{V_{12}}{2|V_{12}|} \left[1 \pm \frac{V_{11}-V_{22}}{\sqrt{(V_{11}-V_{22})^2 + 4|V_{12}|^2}} \right] \right\}^{1/2},$$

$$C_2^{(0)} = \pm \left\{ \frac{V_{12}}{2|V_{12}|} \left[1 \mp \frac{V_{11}-V_{22}}{\sqrt{(V_{11}-V_{22})^2 + 4|V_{12}|}} \right] \right\}^{1/2}, \tag{33}$$

where the index in brackets corresponds to the order of the perturbation theory. Matrix elements of the perturbation (30) $V(i,k=1,2)$ are calculated by using of functions (29) of the degenerate state of the unperturbed Hamiltonian. Taking into account expressions (29) we shall calculate the matrix elements:

$$V_{11} = l\int_o^\pi \psi_n^*(\varphi)\psi_n(\varphi)\cos 2\varphi d\varphi = 0, \quad V_{22} = 0, \tag{34}$$

$$V_{12} = l\int_o^\pi \psi_n^2(\varphi)\cos 2\varphi d\varphi = \begin{cases} 0 & if \quad n \neq 1 \\ \dfrac{l\pi}{4} & if \quad n = 1 \end{cases}. \tag{35}$$

After substitution of those matrix elements in the expressions (33) for the eigenvalues and exact eigenfunctions we shall obtain

$$E_{\pm}^{(1)} = \pm \frac{l\pi}{4}, \quad \psi_{n=1}^{+} = \cos\varphi, \quad \psi_{n=1}^{-} = -i\sin\varphi. \tag{36}$$

Thus the exact wave functions (36) of the undegenerate states only for $n=1$ coincide with the Mathieu function in the limit (27) ($l \to 0$).

The perturbation $V(l,\varphi)$ removes degeneration only for the state $n=1$. Therefore it is only for the state $n=1$ that the spectrum branching occurs at the point $l=0$, which agrees with numerical calculations given in the form of diagrams [12]. It can be assumed that in the case of diminishing l, the merging of energy terms for states $n \neq 1$ takes place at the point at which the states are still defined by the Mathieu functions and not by their limiting values (27). Wave functions for degenerate states $l \neq 0, n \neq 1$ can be composed of the Mathieu functions by using the same arguments as have been used above in composing the wave functions for $l \to 0$ [Eq. (28)]. As a result, we obtain

$$\psi_+^{2m+1} = \psi_n(l,\varphi) = \frac{\sqrt{2}}{2}(ce_n\varphi \pm ise_n\varphi), \quad n = 2m+1, \tag{37}$$

$$\psi_\pm^{2m+2} = \psi_n^*(l,\varphi) = \frac{\sqrt{2}}{2}(ce_n\varphi \pm ise_n\varphi),$$

$$n \neq 1, \quad l \neq 0, \quad n = 2m + 2. \tag{38}$$

Let us assume that at $l_n = l_-^{(n)}$ the removal of degeneration for the nth energy term happens. Then, accordingly, on the left and right from $l_-^{(n)}$ the Mathieu-Schrodinger equation (14)–(17) is possible to rewrite as

$$\hat{H}(l_n)\psi = E(l_n)\psi, \quad l_-^{(n)} \leqslant l_n,$$

$$\hat{H}(l_n)\psi = E(l_n)\psi, \quad l_-^{(n)} \geqslant l_n. \tag{39}$$

Since $\hat{H}$ depends on l continuously [Eq. (14)] then in immediate proximity to the points l_n it is possible to write

$$\hat{H}(l_n > l_-^{(n)}) \approx \hat{H}(l_n < l_-^{(n)}) + \frac{\partial \hat{H}}{\partial l_n}\delta l_n, \tag{40}$$

where δl_n is the infinitesimal area close to l_n. With account of Eq. (14) it is possible to write

$$\frac{\partial \hat{H}}{\partial l_n}\delta l_n = -\delta l_n \cos 2\varphi. \tag{41}$$

Substituting in Eq. (40) $\hat{H}(l_n < l_-^{(n)}) \rightarrow \hat{H}$ and $\hat{H}(l_n > l_-^{(n)}) \rightarrow \hat{H}_o$, the Hamiltonian near the point $l = l_n$ we shall present in the form

$$\hat{H} = \hat{H}_o + \hat{V}(\delta l, \varphi), \tag{42}$$

where

$$\hat{H}_o = \frac{\partial^2}{\partial \varphi^2} - l \cos 2\varphi, \tag{43}$$

$$\hat{V}(\delta l, \varphi) = -\delta l \cos 2\varphi, \quad \delta l > 0. \tag{44}$$

Here for brevity we replaced δl_n by δl.

Let us write the perturbation matrix elements (44) for odd degenerate states (37)

$$V_{\pm\mp}^{2m+1} = \frac{1}{\pi}\int_o^\pi \psi_\pm^{2m+1} V(\delta l, \varphi)(\psi_\mp^{2m+1})^* d\varphi$$

$$= \frac{\delta l}{\pi}\int_o^\pi (ce_{2m+1}^2 - se_{2m+1}^2 \pm i2ce_{2m+1}se_{2m+1}) \times \cos 2\varphi d\varphi, \tag{45}$$

$$V_{\pm\pm}^{2m+1} = \frac{1}{\pi}\int_o^\pi \psi_\pm^{2m+1} V(\delta l, \varphi)(\psi_\pm^{2m+1})^* d\varphi$$

$$= \frac{\delta l}{\pi}\int_o^\pi (ce_{2m+1}^2 + se_{2m+1}^2)\cos 2\varphi d\varphi. \tag{46}$$

Here for brevity we write Mathieu functions without arguments φ and l. Moreover, as the Mathieu functions for the small δl as a result of continually depending from the parameter l changes slightly (see Fig. 1), at the calculation of matrix elements we neglect this dependence.

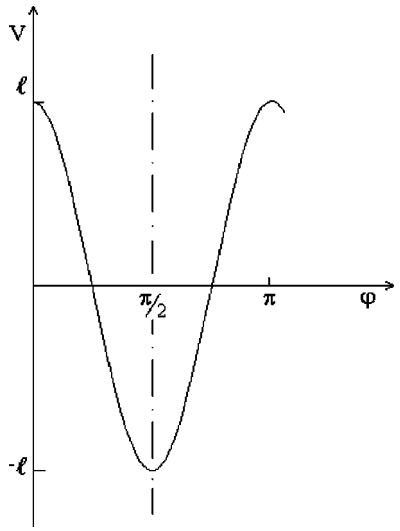

FIG. 1. Plots of the Mathieu functions $ce_4(l+\delta l, \varphi)$, $ce_4(l, \varphi)$, at values of parameters near the point of branching, $l = 2$, $\delta l = 1$, plotted by the use of numerical methods. It is obvious that the small variation of the parameter l leads to the small variation of the Mathieu functions.

To calculate the matrix elements (46) and (47) we use formulas of expansion of Mathieu functions into Fourier series [8],

$$ce_{2m+1} = \sum_{r=0}^{\infty} A_{2r+1}^{2m+1} \cos(2r+1)\varphi, \tag{47}$$

$$se_{2m+1} = \sum_{r=0}^{\infty} B_{2r+1}^{2m+1} \sin(2r+1)\varphi. \tag{48}$$

The factors of expansion A_{2r+1}^{2m+1} and B_{2r+1}^{2m+1} are defined with the help of well-known recursion relations [8,11,12]. Substituting Eqs. (47) in Eqs. (45) and (46), after simple integration, having omitted superscripts for simplicity, one can obtain

$$V_{+-} = V_{-+} = \frac{\delta l}{4}\sum_{r=0}^{\infty} \{A_{2r+1}(A_{2r+3} + A_{2r-1}) - B_{2r+1}(B_{2r+3} + B_{2r-1})\}, \tag{49}$$

$$V_{++} = V_{--} = \frac{\delta l}{4}\sum_{r=0}^{\infty} \{A_{2r+1}(A_{2r+3} + A_{2r-1}) + B_{2r+1}(B_{2r+3} + B_{2r-1})\}. \tag{50}$$

These expressions can be simplified with the help of the recursion relations [8] for A_{2r+1},

$$2[a_{2m+1} - (2r+1)^2]A_{2r+1}^{2m+1} - l_-^{2m+1}(A_{2r-1}^{2m+1} + A_{2r+3}^{2m+1}) = 0, \tag{51}$$

$$2[b_{2m+1}-(2r+1)^2]B_{2r+1}^{2m+1}-l_-^{2m+1}(B_{2r-1}^{2m+1}+B_{2r+3}^{2m+1})=0, \tag{52}$$

where $a_{2m+1}=a_{2m+1}(l)$ and $b_{2m+1}=b_{2m+1}(l)$ are energy terms (Mathieu characteristics [11]) in the nondegenerate area for the states ce_{2m+1} and se_{2m+1} accordingly. In the degenerate area the terms a_{2m+1} and b_{2m+1} converge. Determining from Eqs. (51) and (52), $A_{2r-1}+A_{2r+3}$ and $B_{2r+1}+B_{2r+3}$, and substituting in Eqs. (49) and (50), we obtain

$$V_{++}^{2m+1}=V_{--}^{2m+1}=\frac{\delta l}{l_-^{2m+1}}\left[\frac{a_{2m+1}+b_{2m+1}}{2}-\frac{1}{2}(\tilde{A}^{2m+1}+\tilde{B}^{2m+1})\right], \tag{53}$$

$$V_{-+}^{2m+1}=V_{+-}^{2m+1}=\frac{\delta l}{l_-^{2m+1}}\left[\frac{a_{2m+1}-b_{2m+1}}{2}-\frac{1}{2}(\tilde{A}^{2m+1}-\tilde{B}^{2m+1})\right], \tag{54}$$

where

$$\tilde{A}^{2m+1}=\sum_{r=0}^{\infty}(2r+1)^2[A_{2r+1}^{2m+1}]^2,\quad \tilde{B}^{2m+1}=\sum_{r=0}^{\infty}(2r+1)^2[B_{2r+1}^{2m+1}]^2. \tag{55}$$

For the deriving of the formulas (53) and (54) we used the relations [8,11]

$$\sum_{r=0}^{\infty}[A_{2r+1}]^2=1 \text{ and} \sum_{r=0}^{\infty}[B_{2r+1}]^2=1 \tag{56}$$

and for brevity we wrote the superscript without brackets $(l_-^{(2m+1)}\rightarrow l_-^{2m+1})$.

Substituting the matrix elements (53) and (54) in the expressions of perturbation theory in the approximation of first order with respect to energy eigenvalues and for exact functions of zero approximation (31)–(33) we get

$$E_+=\frac{\delta l}{l_-^{2m+1}}[a_{2m+1}(l_-^{2m+1})-\tilde{A}^{2m+1}(l_-^{2m+1})],$$

$$E_-=\frac{\delta l}{l_-^{2m+1}}[b_{2m+1}(l_-^{2m+1})-\tilde{B}^{2m+1}(l_-^{2m+1})], \tag{57}$$

$$\psi_+^{2m+1}(l_-^{2m+1},\varphi)=ce_{2m+1}(l_-^{2m+1},\varphi),$$

$$\psi_-^{2m+1}(l_-^{2m+1},\varphi)=ise_{2m+1}(l_-^{2m+1},\varphi). \tag{58}$$

After equating the corrections of energy $E_+=E_-$ [Eq. (57)] and taking into account $a_{2m+1}(l_-^{2m+1})=b_{2m+1}(l_-^{2m+1})$, we obtain the equations defining branching points,

$$\tilde{A}^{2m+1}(l_-^{2m+1})=\tilde{B}^{2m+1}(l_-^{2m+1}). \tag{59}$$

According to Eq. (55), both sides of Eq. (59) depend on the coefficients of Fourier expansions of the Mathieu functions (47) and (48), which in their turn depend on l. Equation (59) can be solved only by numerical methods (Table II).

The exact functions of the zero order (58) accurate to the insignificant phase multiplier coincide with the appropriate pair of wave functions (19) from the states of the undegenerate area. So in points l_-^{2m+1} the wave functions of the degenerate states turn into the wave functions of the undegenerate states.

TABLE II. The results of numerical calculations for the coordinates of left branching points.

	Left points of branching	
n	l_-^{2n+2} (48)	l_-^{2n+1} (45)
0	0.30	0
1	2.0	1.2
2	8.0	4.5
3	28.0	13.0

Similarly it is possible to calculate the matrix elements for even states $V_{\pm\pm}^{2m}$ and $V_{\pm\mp}^{2m}$, taking into account expansion formulas of functions $ce_{2m}(l,\varphi)$ and $se_{2m}(l,\varphi)$ in the Fourier series [8,11] and also by use of the similar to Eqs. (51) and (52) relations of recursion. Omitting mathematical details of these calculations, we present final results for approximation of first order with respect to energy eigenvalues and for exact wave functions of zero approximation:

$$E_+=\frac{\delta l}{l_-^{2m}}[a_{2m}(l_-^{2m})-\tilde{A}^{2m}(l_-^{2m})],$$

$$E_-=\frac{\delta l}{l_-^{2m}}[b_{2m}(l_-^{2m})-\tilde{B}^{2m}(l_-^{2m})], \tag{60}$$

$$\psi_+^{2m}(l_-^{2m},\varphi)=ce_{2m}(l_-^{2m},\varphi),$$

$$\psi_-^{2m}(l_-^{2m},\varphi)=ise_{2m}(l_-^{2m},\varphi). \tag{61}$$

The points of a branching l_-^{2m} of energy terms are obtained (by using numerical methods, see Table II) with the help of an equation which can be obtained by means of equating corrections of energy terms (60), $E_+=E_-$:

$$\tilde{A}^{2m}(l_-^{2m})=\tilde{B}^{2m}(l_-^{2m}). \tag{62}$$

Exact wave functions of the zero order (61), accurate to a nonstationary phase factor coincide with the appropriate pair of functions from Eq. (19), describing the nondegenerate states. In other words, in the point l_-^{2m} the removal of degeneration happens.

On the basis of obtained results it is possible to present a qualitative picture of the variation of energy terms on the plane (E,l) in the left-hand area from the line of the separatrix (Fig. 2).

At the end of this section we shall remark that the degenerate state, located at the left of the separatrixes line, can be considered as an analog of classical rotary motion.

B. Degenerate states at major *l*: Area on the right of the separatrix line

With the increasing of l the particle can be trapped in a deep potential well ($V=l\cos 2\varphi, 0<\varphi<\pi$, Fig. 3), perform-

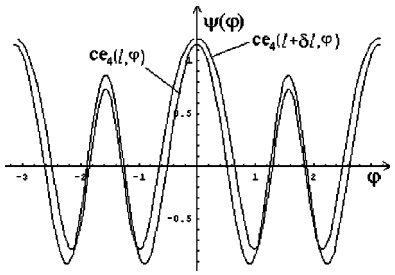

FIG. 2. The energy levels as a function of parameter l on the plane (E,l) on the left area from the separatrix line. The points of the branching of curves represent the boundaries between degenerate and nondegenerate states.

ing oscillatory motion. Properties of wave functions of the quantum oscillator near the bottom of the well are well known. This is the alternation of even and odd wave functions relative to the center of the potential well $\pi/2$ and the presence of zeros of wave functions. With the help of the third column of Table I it is possible to write symmetry conditions close to $\pi/2$:

$$ce_m\left(\frac{\pi}{2}+\varphi\right)=(-1)^m ce_m\left(\frac{\pi}{2}-\varphi\right),$$

$$se_m\left(\frac{\pi}{2}+\varphi\right)=(-1)^{m+1} se_m\left(\frac{\pi}{2}-\varphi\right), \tag{63}$$

i.e., $ce_{2m}(\varphi)$, $se_{2m+1}(\varphi)$ are even functions and $se_{2m}(\varphi)$, $ce_{2m+1}(\varphi)$ are odd functions. Functions $ce_{2m}(\varphi)$, $se_{2m+1}(\varphi)$, $ce_{2m+1}(\varphi)$, and $se_{2m+2}(\varphi)$ have m real zeros between $\varphi=0$ and $\varphi=\pi/2$ (not considering zeros on edges).

The existing alternation of states (Fig. 2) in the area along the line of the separatrix is conditioned by the properties of states at the small l. With the help of the expressions (63) it is possible to determine easily that in the spectrum of the states along the line $E=l$ two (instead of one) even states alternate with two odd states, and so on. To get the alternation, caused now by properties at major l, two even states must degenerate in one even, and two odd in one odd. So we

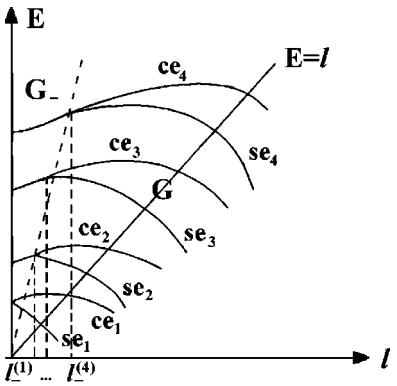

FIG. 3. The dependence of interaction energy V on phase φ. The interaction has the following properties of symmetry: 1. $V(\varphi)=V(-\varphi)$; 2. $V(\varphi)=V(\pi+\varphi)$; 3. $V(\varphi)=V(\pi-\varphi)$.

come to the conclusion that with increasing l two levels with wave functions $ce_{2m}(\varphi)$ and $se_{2m+1}(\varphi)$ coming nearer amalgamate in one level, and other two levels, $ce_{2m}(\varphi)$ and $se_{2m+1}(\varphi)$, also in one level. The levels obtained in this way will be doubly degenerated. It can be assumed that with the growth of l the states defined by the symmetry group G transform to the states with the symmetry of an invariant subgroup G_+ [Eq. (24)]. This transformation takes place at the merging point of nondegenerate terms $l=l_+^{(n)}$. Recall that subgroup G_+ contains two elements: the unit element e and the reflection element with respect to the symmetry center of the well $b=G(\varphi\to\pi-\varphi)$. Complex wave functions of the area of degenerate states, with the symmetry of the invariant subgroup G_+, can be composed of pairs of functions of merged states in the same manner as we have done above for the area of small l for states with the symmetry of G_-.

Not iterating these reasons, we shall write complex wave functions corresponding to the degenerated states in the form

$$\xi_{2m}^{\pm}(\varphi)=ce_{2m}(\varphi)\pm ise_{2m+1}(\varphi),\ \text{even states}, \tag{64}$$

$$\zeta_{2m+1}^{\pm}(\varphi)=ce_{2m+1}(\varphi)\pm ise_{2m+2}(\varphi),\ \text{odd states}. \tag{65}$$

In the base of complex wave functions $\xi_{2m}^{\pm}$ and $\zeta_{2m+1}^{\pm}$ the indecomposable representation of the subgroup G_+ is realized. Parity of the wave functions $\xi_{2m}^{\pm}$ and $\zeta_{2m+1}^{\pm}$ with respect to the transformations of the subgroup G_+ characterizes an important property of wave functions evenness of the quantum oscillatory process.

Let us set about with the calculation of the matrix elements of interaction (44) for the states given by the wave functions (64):

$$W_{\pm\mp}^{2m}=\frac{\delta l}{\pi}\int_o^{2\pi}\xi_{2m}^{\pm}\cos 2\varphi(\zeta_{2m}^{\mp})^* d\varphi \tag{66}$$

and

$$W_{\pm\pm}^{2m}=\frac{\delta l}{\pi}\int_o^{2\pi}\xi_{2m}^{\pm}\cos 2\varphi(\xi_{2m}^{\pm})^* d\varphi,\quad \delta l<0. \tag{67}$$

Let us use expansion formulas in the Fourier series [8] for the Mathieu functions with even index

$$ce_{2m}=\sum_{r=0}^{\infty}A_{2r}^{2m}\cos 2r\varphi,$$

$$se_{2m}=\sum_{r=0}^{\infty}B_{2r}^{2m}\sin 2r\varphi \tag{68}$$

and by recurrence relations

$$(E-4r^2)A_{2r}-\delta l(A_{2r-2}+A_{2r+2})=0,\quad E=a_{2m}(l),$$

$$r=2,3,\ldots,$$

$$[E-(2r+2)^2]B_{2r}-\delta l(B_{2r}+B_{2r+4})=0,\quad E=b_{2m+2}(l),$$

TABLE III. The results of numerical calculations for the coordinates of right branching points.

	Right points of branching	
n	l_+^{2n}	l_+^{2n+1}
1	15.0	17.5
2	25.0	32
3	41	51

$$r = 1,2, \ldots . \tag{69}$$

After simple calculations similar to those in the previous section, we shall get

$$W_{++}^{2m} = W_{--}^{2m} = \frac{1}{2}(a_{2m} + b_{2m}) - \frac{1}{2}(A_o^2 + \tilde{A}^{2m} + \tilde{B}^{2m}), \tag{70}$$

$$W_{+-}^{2m} = W_{-+}^{2m} = \frac{1}{2}(a_{2m} - b_{2m}) - \frac{1}{2}(A_o^2 + \tilde{A}^{2m} + \tilde{B}^{2m}), \tag{71}$$

where

$$\tilde{A}^{2m} = \sum_{r=1}^{\infty} (2r)^2 [A_{2r}^{2m}]^2, \quad \tilde{B}^{2m} = \sum_{r=0}^{\infty} (2r)^2 [B_{2r}^{2m}]^2. \tag{72}$$

For deriving of the formulas (70) and (71) we have used the relations [8,11]

$$2[A_o]^2 + \sum_{r=1}^{\infty} [A_{2r}]^2 = 1 \text{ and } \sum_{r=0}^{\infty} [B_{2r+1}]^2 = 1. \tag{73}$$

Substituting the matrix elements (70) and (71) in the formulas of the secular perturbation theory for the first order terms of the energy eigenvalues and for exact functions of the zero approximation (33) we shall obtain

$$E_+^{2m} = \frac{\delta l}{l_+^{2m}}[a_{2m}(l_+^{2m}) - A_o^2 - \tilde{A}^{2m}],$$

$$E_-^{2m+1} = \frac{\delta l}{l_+^{2m}}[b_{2m+1}(l_+^{2m}) - \tilde{B}^{2m+1}], \tag{74}$$

$$\psi_1^{(o)} = ce_{2m}(\varphi), \psi_2^{(o)} = i\ se_{2m+1}(\varphi). \tag{75}$$

After equating the corrections of energies (74) $E_+ = E_-$ and taking into account the fact that at the point of merging of terms $a_{2m}(l_+^{2m}) = b_{2m+1}(l_+^{2m})$, we obtain the equation for finding l_+^{2m}:

$$A_o^2 + \tilde{A}^{2m} = \tilde{B}^{2m+1}. \tag{76}$$

Equation (76), like Eq. (62), can be solved only by numerical methods (see Table III).

The similar calculations can be easily done for the odd states $\xi_\pm^n$ [Eq. (65)]. As follows from these calculations, at particular values l the degeneration is removed. The results, obtained in this section, are plotted in Fig. 4.

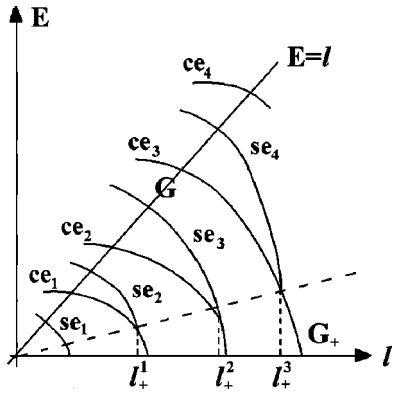

FIG. 4. The energy levels as a function of the parameter l on the plane (E,l) on the area to the right of the separatrix line. The points of the branching of curves represents points of degeneration of terms in this area.

Figures 2 and 4 supplement each other: in the field of intersection with the separatrix the curves of the Figs. 2 and 4 are smoothly joined.

So, we shall add up outcomes obtained in this section. The Mathieu-Schrodinger equation has an appointed symmetry. The transformations of the symmetry of the Mathieu functions form group G, which is isomorphic to the quaternary group of Klein. To this symmetry on a plane (E,l) corresponds the appointed area along the line of the separatrix $E=l$, containing nondegenerated energy terms. This area is restricted double sided by the areas of degenerate states, which are characterized by the symmetry properties of the invariant subgroups G_- and G_+, respectively. The boundaries of these areas are defined by the branching points of energy terms existing both on the right and on the left of the separatrix. Equations for determining the branching points of energy terms are obtained. The equations are solved only numerically.

The area of degenerate states is the quantum-mechanical analogs of two forms of motion of the classical mathematical pendulum—rotary and oscillatory. Comparing results of quantum reviews with classical, we remark that these two conditions of motion at quantum reviewing are divided by the area of a finite measure, whereas at the classical reviewing measure the separatrix is equal to zero.

V. INTEGRALS OF MOTION: AVERAGE VALUES OF SOME OBSERVABLE QUANTITIES

Let us find out the complete set of physical quantities for our system. For this purpose it is necessary to write all transformations which commute with the Hamiltonian (14). As we already have established (see Table I), these transformations form a quaternary group of Klein. In this paragraph we compare to each element of this group the appropriate quantum-mechanical operators producing these transformations. So, the element of group a is the operator of inversion $\hat{I}_o[\hat{I}_o\psi(\varphi) = \psi(-\varphi)]$, which commutes with the Hamiltonian $\hat{H}$ [Eq. (14)],

$$\hat{H}\hat{I}_o - \hat{I}_o\hat{H} = 0. \tag{77}$$

TABLE IV. Table of energy terms and quantum numbers for the eigenfunctions of nondegenerate states ($l>0$).

	E_0	I_0	$I_{\pi/2}$	T_π
$ce_{2m}(l,\varphi)$	$a_{2m}(l)$	1	1	1
$ce_{2m+1}(l,\varphi)$	$a_{2m+1}(l)$	1	−1	−1
$se_{2m+1}(l,\varphi)$	$b_{2m+1}(l)$	−1	1	−1
$se_{2m+2}(l,\varphi)$	$b_{2m+2}(l)$	−1	−1	1

The eigenfunction of the Hamiltonian $\psi(\varphi)$ also is the eigenfunction for the operator of the inversion $\hat{I}_o$,

$$\hat{I}_o\psi(\varphi)=I_o\psi(\varphi). \tag{78}$$

Acting once again on Eq. (78) by means of operator $\hat{I}_o$ one can obtain $\hat{I}_2^o\psi(\varphi)=I_o^2\psi(\varphi)=\psi(\varphi)$. It follows that eigenvalues of the operator of inversion are $I_o=\pm1$. Thus the eigenfunctions have fixed parity, which remains invariable in time. The element b of the Klein symmetry group, also commuting with the Hamiltonian, is the operator of inversion relative to the center of the potential well (Fig. 3)—$\hat{I}_{\pi/2}$. It is possible to show similarly that the relevant eigenvalues of the operator are $\hat{I}_{\pi/2}=\pm1$. The element c of the Klein symmetry group, commuting with the Hamiltonian, is the operator of translation $\hat{T}_\pi$ with respect to the phase on the distance π. The eigenvalues of the operator of translation $\hat{T}_\pi$ are $T_\pi=\pm1$.

In the area of nondegenerate states designated in Figs. 2 and 4 by means of G, all four operators form the compete set: the energy $\hat{H}$, inversion $\hat{I}_o$, inversion concerning the center of the well $\hat{I}_{\pi/2}$, and translation with respect to the phase $\hat{I}_\pi$ have the same eigenfunctions. For example to the eigenstate $ce_{2n+1}(l)$ corresponds the energy term $a_{2n+1}(\varphi)$, $I_o=1$, $I_{\pi/2}=-1$, $T_\pi=-1$. In Table IV we reduce quantum numbers for eigenfunctions for the undegenerate states.

In the degenerated area designated by G_- in Fig. 1, where the states are characterized by wave functions $\psi_{2m}^{\pm}$ and $\psi_{2m+1}^{\pm}$ [Eq. (36)], operators of symmetry produce transformations

$$\hat{I}_o\psi_{2m}^{\pm}=\psi_{2m}^{\mp},\quad \hat{I}_o\psi_{2m+1}^{\pm}=\psi_{2m+1}^{\mp},$$

$$\hat{I}_{\pi/2}\psi_{2m}^{\pm}=\psi_{2m}^{\mp},\quad \hat{I}_{\pi/2}\psi_{2m+1}^{\pm}=-\psi_{2m+1}^{\mp}, \tag{79}$$

$$\hat{T}_\pi\psi_{2m}^{\pm}=\psi_{2m}^{\pm},\quad \hat{T}_\pi\psi_{2m+1}^{\pm}=-\psi_{2m+1}^{\pm}. \tag{80}$$

According to the relations (79), the wave functions $\psi_{2m}^{\pm}$ and $\psi_{2m+1}^{\pm}$ are not eigenfunctions for the inversion operators $\hat{I}_o$ and $\hat{I}_{\pi/2}$, but are eigenfunctions for the Eq. (80) operator of translation $\hat{T}_\pi$.

This result is easy to understand because degenerate area G_- corresponds to the rotary motion. Since the rotary motion is valid for the sufficient high energies $E>l_->l$, properties of the symmetry (Fig. 3), defining the properties of the system at the inversion, are unessential. In this area translation symmetry leading to the infinite motion, i.e., periodic recurrence, plays an essential role.

From relations (79) and (80) it follows that at the passage from area G into area G_- the destruction of two integrals of motion [I_o and $\hat{I}_{\pi/2}$, Eq. (79)] happens, but the other two—energy and $\hat{T}_\pi$—are maintained [Eq. (80)].

In degenerate area G_+ (Fig. 1), in which the states are characterized by wave functions $\xi_n^{\pm}$ and $\zeta_n^{\pm}$ [Eqs. (64) and (65)], the operators of the symmetry produce transformations

$$\hat{I}_o\xi_{2m}^{\pm}=\xi_{2m}^{\mp},\quad \hat{I}_o\zeta_{2m+1}^{\pm}=\zeta_{2m+1}^{\mp},$$

$$\hat{T}_\pi\xi_{2m}^{\pm}=\xi_{2m}^{\mp},\quad \hat{T}_\pi\zeta_{2m+1}^{\pm}=-\zeta_{2m+1}^{\mp}, \tag{81}$$

$$\hat{I}_{\pi/2}\xi_{2m}^{\pm}=\xi_{2m}^{\pm},\quad \hat{I}_{\pi/2}\zeta_{2m+1}^{\pm}=-\zeta_{2m+1}^{\pm}. \tag{82}$$

From relations (81) and (82) also follows that at the passage from area G into area G_+ the destruction of two integrals of motion [$\hat{I}_o$ and $\hat{T}_\pi$, Eq. (81)] happens, but the other two—energy and $\hat{I}_{\pi/2}$ [Eq. (82)]—are maintained.

It is clear from relations (81), that the wave functions $\xi_n^{\pm}$ are not eigenfunctions for the operators I_o and $\hat{T}_\pi$. The functions $\xi_n^{\pm}$ are the eigenfunctions for the operator of inversion with respect to the symmetry axis of the potential well—$\hat{T}_{\pi/2}$ (Fig. 3). This result can be understood by assuming that the degenerate area $G_+(E<l_+<l)$ corresponds to the oscillatory motion performed by the particle captured in the potential well. Because the action of operators $\hat{I}_o$ and $\hat{T}_\pi$ transfers the particle to the other "potential wells" (i.e., hinder the capture in one of the potential wells), determining by them properties of the system in case of oscillatory motion will be inessential. The main role in area G_+ takes the symmetry relatively to the center of the potential hole $\hat{I}_{\pi/2}$, describing parity of the oscillatory states.

Let us proceed to the computation of some physical quantities characterizing the system. Our interest will be fixed on the computation of the mean of the action variation ΔI and its square $(\Delta I)^2$:

$$\langle\Delta\hat{I}\rangle=-i\frac{\hbar}{2}\left\langle\psi\left|\frac{\partial}{\partial\varphi}\right|\psi\right\rangle,\quad \langle(\Delta\hat{I})^2\rangle=-\frac{\hbar^2}{4}\left\langle\psi\left|\frac{\partial^2}{\partial\varphi^2}\right|\psi\right\rangle, \tag{83}$$

where

$$\langle\psi|\hat{A}|\psi\rangle=\frac{1}{\pi}\int_o^{2\pi}\psi(\varphi)A\psi^*(\varphi)d\varphi.$$

The mean calculated with the help of different wave functions of the system, for different areas of the plane (E,l), naturally will be different. Let us begin with the case of the free rotation ($l=0$). With the help of the wave functions (29) we shall obtain

$$\left\langle\psi_{1,2}\left|\frac{\partial}{\partial\varphi}\right|\psi_{1,2}\right\rangle=\pm in. \tag{84}$$

Two signs in Eq. (84) correspond to rotation in two opposite directions. The wave functions $\psi_{1,2}$ are the eigenfunctions all at the same time for the Hamiltonian (25) and for the operator $\partial/\partial\varphi$. Therefore for these states the eigenvalues and the average values coincide. For the computation of the mean the operator $\partial^2/\partial\varphi^2$ suffices to raise to the square power expression (84).

For area G_- the wave functions can be written down in the form (37) and (38). In this area $\partial/\partial\varphi$ does not commute with the Hamiltonian (14) and the appropriate values are not precisely measurable. For the average value $\partial/\partial\varphi$ with the use of wave functions (37) and expansion formulas in the Fourier series (47) and (48) we shall get

$$\left\langle \psi^{\pm}_{2n+1} \left| \frac{\partial}{\partial \varphi} \right| \psi^{\pm}_{2n+1} \right\rangle = \pm i\overline{A^{2n+1}B^{2n+1}}, \tag{85}$$

where

$$\overline{A^{2n+1}B^{2n+1}} = \sum_{r=0}^{\infty} (2r+1)A_{2r+1}B_{2r+1}.$$

Two signs before the sum correspond to the different directions of rotation. For the calculation of the average value $\partial^2/\partial\varphi^2$ we shall take advantage of the energy integral with the help of which it is possible to produce the replacement $\partial^2/\partial\varphi^2 \to -(E_o - l\cos 2\varphi)$. In view of it we shall get

$$\left\langle \psi^{\pm}_{2m+1} \left| \frac{\partial^2}{\partial \varphi^2} \right| \psi^{\pm}_{2m+1} \right\rangle = -E_o + V_{\pm\pm}. \tag{86}$$

Substituting in Eq. (86) the results of the previous evaluations for the V_{++} [Eq. (50)], we shall obtain

$$\left\langle \psi^{\pm}_{2m+1} \left| \frac{\partial^2}{\partial \varphi^2} \right| \psi^{\pm}_{2m+1} \right\rangle = -\frac{1}{2}[\tilde{A}^{2m+1} + \tilde{B}^{2m+1}]. \tag{87}$$

As was expected, the square of the mean (85) does not coincide with the mean of square (87).

Let us calculate the average values of quantities for the states of undegenerate area G. With the help of wave functions (19) we obtain

$$\left\langle \psi_n \left| \frac{\partial}{\partial \varphi} \right| \psi_n \right\rangle = \frac{1}{\pi}\int_o^{2\pi} \psi_n(\varphi)\frac{\partial}{\partial \varphi}\psi_n^*(\varphi)d\varphi = 0. \tag{88}$$

Having taken advantage of the energy integral (14) for computation of the mean of $\partial^2/\partial\varphi^2$ for the system in the state $\psi_{2n+1} = se_{2n+1}(\varphi)$, we shall get

$$\left\langle se_{2n+1} \left| \frac{\partial^2}{\partial \varphi^2} \right| se_{2n+1} \right\rangle = -E_n + \frac{l}{\pi}\int_o^{2\pi} se^2_{2n+1}\cos 2\varphi d\varphi$$

$$= -\tilde{B}^{2n+1}. \tag{89}$$

And at last, with the help of the states $\xi^{\pm}_m$ of the degenerate area G_+ for the mean we shall obtain

TABLE V. Some average values of the action deviation and its square for the states corresponding to the different areas of the plane (E,l).

	$l \ll E_0$ $l=0$	$l < l_- < E_0$ G_-	$l_- < l < l_+$ G	$E_0 < l_+ < l$ G_+
$\langle \Delta I \rangle_n$	$\mp\frac{\hbar}{2}n$	$\mp\frac{\hbar}{2}\overline{A^{2m+1}B^{2m+1}}$	0	0
$\langle (\Delta I)^2 \rangle_n$	$\frac{\hbar}{2}n^2$	$\frac{\hbar^2}{4}\frac{\tilde{A}^{2m+1}\tilde{B}^{2m+1}}{2}$	$\frac{\hbar^2}{4}\tilde{B}^{2m+1}$	$\frac{\hbar^2}{4}\frac{A_0^2\tilde{A}^{2m} + \tilde{B}^{2m+1}}{2}$

$$\left\langle \xi^{\pm}_m \left| \frac{\partial}{\partial \varphi} \right| \xi^{\pm}_m \right\rangle = 0 \tag{90}$$

and

$$\left\langle \xi^{\pm}_{2m} \left| \frac{\partial^2}{\partial \varphi^2} \right| \xi^{\pm}_{2m} \right\rangle = -E_m + \frac{1}{\pi}\int_o^{2\pi} [\xi^{\pm}_{2m}(\varphi)]^2 \cos 2\varphi d\varphi$$

$$= -\tfrac{1}{2}[A_o^2 + \tilde{A}_{2m} + \tilde{B}_{2m+1}]. \tag{91}$$

With the help of expressions (83) and the results obtained in this section [Eqs. (84)–(91)], we can compose the table of mean value variations for the action's deviation mean and its square with the increase of l (Table V).

As follows from Table V, the relation $\langle \Delta I \rangle_n^2 = \langle (\Delta I)^2 \rangle_n$ is fulfilled only for the values of the column $l \ll E_o$. This is understandable if we recall that it is only in this case that the system is in the eigenstate of the operator $\Delta I \sim \partial/\partial\varphi$ and therefore its eigenvalues are defined precisely. In all other cases the system is not in the eigenstate of the operator ΔI and therefore for the columns G_-, G, and G_+ we have $\langle \Delta I \rangle_n^2 \neq \langle (\Delta I)^2 \rangle_n$.

Using Table V, we can observe how $\langle \Delta I \rangle_n$ diminishes to zero with the growth of l. This is understandable if we recall that the value $\langle \Delta I \rangle_n$ is proportional to the rotational component of motion. Indeed, in the first column ($l=0$) of Table IV the value of $\langle \Delta I \rangle_n$ corresponds to the free motion mode, while in the second column ($l < l_-$) the value of $\langle \Delta I \rangle_n$ corresponds to the rotation weakened by the influence of a periodic potential. The third and fourth columns ($l_- < l < l_+$ and $l_+ < l$) correspond to the quantum analogs of the separatrix and oscillatory motion, respectively, where the rotational component is totally absent $\langle \Delta I \rangle_n$.

VI. QUANTUM ANALOG OF THE STOCHASTIC LAYER

In the case of Hamiltonian systems performing a finite motion, a stochastic layer formed in a neighborhood of the separatrix under the action of an arbitrary periodic perturbation is a minimal phase space cell that contains the gem of

stochasticity [4]. In this section we shall try to find out what can be considered as the quantum analog of the stochastic layer.

Let us assume that the pumping amplitude is modulated by the slow variable electromagnetic field. The influence of modulation is possible to take into account by means of such replacement in the Mathieu-Schrodinger equation (16),

$$l \to l + \Delta l \cos \nu t,\ \Delta l < l. \tag{92}$$

Here Δl stands for the amplitude of modulation in dimensionless units [see Eq. (18)], ν is the frequency of modulation. We suppose that the slow variation of l can embrace some quantity of the branching points on the left and on the right of the separatrix line (Figs. 1 and 2),

$$\Delta l \geqslant |l_+^n - l_-^n|,\ n = 1,2, \ldots, N. \tag{93}$$

As a result of replacement (92) in the Hamiltonian (14), we get

$$\hat{H} = \hat{H}_o + \hat{H}'(t), \tag{94}$$

$$\hat{H}'(t) = \Delta l \cos 2\varphi \cos \nu t, \tag{95}$$

where $\hat{H}_o$ is the universal Hamiltonian (14) and $\hat{H}'(t)$ is the perturbation appearing as a consequence of the pumping modulation.

It is easy to see that the matrix elements of perturbation (95) $\hat{H}'(t)$ for nondegenerate states are equal to zero. Really, having applied expansion formulas of the Mathieu functions in the Fourier series (47) and (48) it is possible to show

$$\langle ce_n|\hat{H}'(t)|se_n\rangle \sim \int_o^{2\pi} ce_n(\varphi)\cos 2\varphi se_n(\varphi)d\varphi = 0 \tag{96}$$

for even as well as for odd n. The expressions of the selection rules (96) will be fulfilled for values l from the area $l_-^n \leqslant l \leqslant l_+^n$. Transitions between levels cannot be conditioned by the time-dependent interaction (95). It is expedient to include Eq. (95) in the unperturbed part of the Hamiltonian. The Hamiltonian, obtained in such a way, is slowly depending on the parameter l. So, instead of Eqs. (94) and (95) for the nondegenerated area G we get the Hamiltonian in the form

$$\hat{H} = -\frac{\partial^2}{\partial \varphi^2} + l(t)\cos 2\varphi, \tag{97}$$

$$l(t) = l + \Delta l \cos \nu t. \tag{98}$$

There arises the situation in which the system slowly "creeps" along the Mathieu characteristics and, in doing so, encloses the branching points on the left l_-^n or on the right l_+^n.

A. Irreversible "creeping" of energy term populations due to the influence of a measuring arrangement

According to the general rules of quantum mechanics, probabilities that the system will pass to the eigenstate of another area are defined by the coefficients of expansion of the wave function of one area into the eigenfunctions of another area. Let us assume that, initially, the system was in one of the eigenstates from the nondegenerate area G, for example in the state ce_{2n}. After a quarter of the modulation period $T/4$ (where $T=2\pi/\nu$), having passed through the point l_-^n, the system finds itself in the degenerated area G_-. In this case the system will pass to degenerate states $\psi_{2n}^{\pm}$ with probabilities,

$$P(ce_{2n} \to \psi_{2n}^{\pm}) = \left| \frac{1}{\pi} \int_o^{2\pi} ce_{2n}(\varphi)\psi_{2n}^{\pm *}(\varphi)d\varphi \right|^2$$
$$= \frac{1}{2\pi} \left| \int_o^{2\pi} ce_{2n}(\varphi)[ce_{2n}(\varphi) \pm ise_{2n}(\varphi)]^* d\varphi \right|^2 = \frac{1}{2}. \tag{99}$$

For deriving Eq. (99) we used the condition of normalization (20) and orthogonality [8]

$$\int_o^{2\pi} ce_k(\varphi)se_{l+1}(\varphi)d\varphi = 0, \quad l,k = 0,1,2, \ldots . \tag{100}$$

The passage (99) is based on the assumption of having a deep physical sense. As is generally known, in quantum mechanics symmetry with respect to both directions of time is expressed in the invariance of the wave equation with respect to the variation of the sign of time t and simultaneous replacement ψ by way of ψ^*. However, it is necessary to remember that this symmetry concerns only the equations, but not the concept of a measurement playing a fundamental role in the quantum mechanics [10,13]. "Measurement" is understood as the process of interaction of the quantum system with the classical object usually called "instrument." Under the measuring arrangement, consisting of the analyzer and detector, one must not imagine the laboratory's instrument. So, the role of the analyzer plays in our case the modulating field, which is capable to "drag" the system through the branching points. When passing through the branching point from one area to another, the state remains unchanged. However, being an eigenstate in one area, it will not be an eigenstate in another. At the passage through branching points there occurs a spectral expansion of the initial wave function belonging to the region of one symmetry over the eigenfunctions belonging to the region of another symmetry. The presence only of the analyzer reserves a pure state and the process remains reversible. However, further we shall assume the presence of the detector, defining which of the states, ψ_n^+ or ψ_n^-, is involved in passage. The transition of the system to various states defined by probabilities (100) is fixed by means of the action of the detector. The presence of the detector is expressed formally in averaging with respect to phase and neglecting the interference term usually appearing in the expression for a distribution function. As a result of averaging the partial loss of information about the condition of the system takes place and a mixed state is generated.

As it follows from Eq. (99), after the quarter period degenerated rotary states ψ_{2n}^{+} and ψ_{2n}^{-} will be occupied with the identical probability. After the half period $\frac{1}{2}T$ the system again appears in the area G going through the branching point l_{-}^{n} in the reverse direction. In so doing probabilities for the transitions into the state ce_{2n} as well as in se_{2n} will be distinct from zero,

$$P(\psi_{2n}^{\pm} \to ce_{2n}) = \frac{1}{2}\left|\frac{1}{\pi}\int_{o}^{2\pi}[ce_{2n}(\varphi) \pm ise_{2n}(\varphi)]ce_{2n}(\varphi)d\varphi\right|^{2} = \frac{1}{2}, \tag{101}$$

$$P(\psi_{2n}^{\pm} \to se_{2n}) = \frac{1}{2}\left|\frac{1}{\pi}\int_{o}^{2\pi}[ce_{2n}(\varphi) \pm ise_{2n}(\varphi)]se_{2n}(\varphi)d\varphi\right|^{2} = \frac{1}{2}. \tag{102}$$

Here we have used again normalization (20) and orthogonality (100) relations. It is possible to write the transition probability from ce_{2n} in one of the degenerated states $\psi_{2n}^{\pm}$ and back in the ce_{2n},

$$\begin{aligned} P_{-}(ce_{2n} \leftrightarrow ce_{2n}) &\equiv P_{-}(ce_{2n} \to \psi_{2n}^{\pm} \to ce_{2n}) \\ &= P(ce_{2n} \to \psi_{2n}^{+})P(\psi_{2n}^{+} \to ce_{2n}) \\ &\quad + P(ce_{2n} \to \psi_{2n}^{-})P(\psi_{2n}^{-} \to ce_{2n}). \end{aligned} \tag{103}$$

Here the first summand corresponds to the passage through the degenerated state ψ_{2n}^{+} and the second one to the passage through ψ_{2n}^{-}. It is easy to see with the help of previous computations (99), (101), and (102) that contributions of these passages are identical and individually equal to $1/4$. Therefore finally we have

$$P_{-}(ce_{2n} \leftrightarrow ce_{2n}) = \frac{1}{2}. \tag{104}$$

Similarly it may be shown that transition probability from the state ce_{2n} in one of the degenerated states $\psi_{2n}^{\pm}$ and back in the area G, in state se_{2n} by means of going through the point l_{-}^{n} is

$$P_{-}(ce_{2n} \leftrightarrow se_{2n}) = P(ce_{2n} \to \psi_{2n}^{+})P(\psi_{2n}^{+} \to se_{2n}) + P(ce_{2n} \to \psi_{2n}^{-})P(\psi_{2n}^{-} \to se_{2n}) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}. \tag{105}$$

Thus the system being at the initial moment in the eigenstate ce_{2n} at the end of the half period of modulation appears in the mixed state ρ_{2n}, in which the states ce_{2n} and se_{2n} are intermixed with identical weight, and corresponding levels are populated with identical probabilities.

After the expiration of a quarter of the cycle the system will pass from the area G (the state ρ_{2n}) into the area G_{+}, going through the point l_{+}^{n}. In passages from the area G_{+} four states $\xi_{2n}^{\pm}=(1/\sqrt{2})(ce_{2n} \pm ise_{2n+1})$ and $\zeta_{2n-1}^{\pm}=(1/\sqrt{2})\times(ce_{2n-1} \pm ise_{2n})$ take part. So, taking into consideration the above-mentioned probabilities of transitions we get

$$P(\rho_{2n} \to \xi_{2n}^{\pm}) = \frac{1}{4\pi}\left|\int_{o}^{2\pi}[ce_{2n}(\varphi) + se_{2n}(\varphi)] \times [ce_{2n}(\varphi) \mp ise_{2n+1}(\varphi)]d\varphi\right|^{2} = \frac{1}{4}, \tag{106}$$

$$P(\rho_{2n} \to \zeta_{2n}^{\pm}) = \frac{1}{4\pi}\left|\int_{o}^{2\pi}[ce_{2n}(\varphi) + se_{2n}(\varphi)] \times [ce_{2n-1}(\varphi) \mp ise_{2n}(\varphi)]d\varphi\right|^{2} = \frac{1}{4}. \tag{107}$$

For deriving the last expressions in addition to the normalization conditions we have used the orthogonality conditions [8]

$$\int_{o}^{2\pi} ce_{n}(\varphi)ce_{m}(\varphi)d\varphi = \int_{o}^{2\pi} se_{n+1}(\varphi)se_{m+1}(\varphi)d\varphi = 0, \quad m \neq n. \tag{108}$$

On the basis of Eq. (106) and (107) we conclude that after the time $\frac{3}{4}T$ the system will be in the area G_{+} in one of four oscillatory states $\xi_{2n-1}^{\pm}$ and $\zeta_{2n-1}^{\pm}$ with the identical probability equal to $1/4$.

After one cycle T the system gets back in the area G, from which it started the transition from the level ce_{2n}. Upon returning, four levels, ce_{2n}, se_{2n}, ce_{2n-1}, and se_{2n+1}, will be involved. Calculating probabilities of passages from the oscillatory states of the area G_{+} to these four levels we obtain

$$P(\xi_{2n}^{\pm} \to ce_{2n}) = P(\xi_{2n}^{\pm} \to se_{2n+1}) = 1/2, \tag{109}$$

$$P(\xi_{2n-1}^{\pm} \to se_{2n}) = P(\zeta_{2n-1}^{\pm} \to se_{2n-1}) = 1/2. \tag{110}$$

The probability of passages from the nondegenerated area G_{+} to the area in one of the oscillatory states $\xi_{2n}^{\pm}$, $\xi_{2n-1}^{\pm}$ and back in the area G will be

$$P_{+}(\rho_{2n} \leftrightarrow se_{2n}) = P(\rho_{2n} \to \xi_{2n}^{+})P(\xi_{2n} \to se_{2n+1}) + P(\rho_{2n} \to \xi_{2n}^{-})P(\xi_{2n}^{-} \to se_{2n+1}) = \frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{1}{4}. \tag{111}$$

Similarly it is possible to show

$$P_{+}(\rho_{2n} \leftrightarrow ce_{2n}) = P_{+}(\rho_{2n} \to se_{2n}) = P_{+}(\rho_{2n} \to ce_{2n-1}) = 1/4. \tag{112}$$

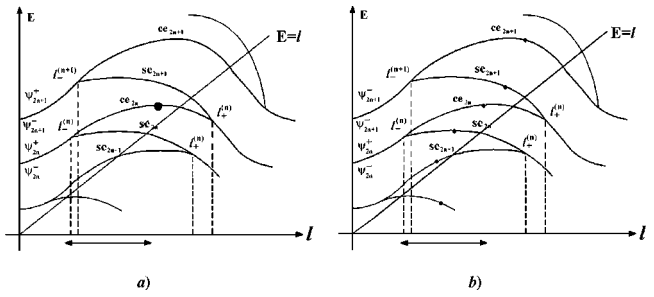

FIG. 5. The fragment of the energy terms, participating in passages calculated in the text. (a) The initial state. The particle is in the state of ce_{2n}. (b) The final state. Levels which are affected by a change of the field amplitude get populated.

Thus after the lapse of time T four levels of the nondegenerate area G will be occupied with the identical probabilities $1/4$ (Fig. 5).

The motion of the system upwards on energy terms will cease upon reaching the level for which the points of the branching in Fig. 4 are on the distance at which the condition (93) no longer is valid. The motion of the system downwards will be stopped upon reaching the zero level. If the system at the initial moment is in the state $2n=N/2$, then after $N/2$ cycles of modulation all N levels will be occupied.

It is easy to calculate a level population for the extremely upper and extremely lower levels. Really, the level population for extreme levels is possible to define with the help of a Markov chain containing only one possible trajectory in the spectrum of Mathieu characteristics:

$$P\left(ce_{N/2},t_o;se_{N/2+1},t_o+\frac{T}{2};\ldots se_N,t_o+N\frac{T}{2}\right)=P\left(se_N,t_o+N\frac{T}{2}\right.$$
$$\left.\leftarrow ce_{N-1},t_o+(N-1)\frac{T}{2}\right)\cdots P\left(ce_{N/2+1},t_o+T\leftarrow se_{N/2+1},t_o\right.$$
$$\left.+\frac{T}{2}\right)P\left(se_{N/2+1},t_o+\frac{T}{2}\leftarrow ce_{N/2},t_o\right), \tag{113}$$

where t_o is an initial time. Here, when discussing the transition probabilities from one state to another, we also use a time argument.

It is possible to write a similar chain of level population for the extremely lower level. As the probabilities of passages, included in the right side of Eq. (113) by way of factors, are equal to 1/2, then probabilities of an extreme level population will be $(1/2)^{N/2}$.

As to the Markovian chain for nonextreme levels, it has a cumbersome form and we do not give it here.

Let us note that here irreversibility is conditioned by the interaction on the system of detector in moments of time overcoming branching points $t_o+nT/2;(n=1,2,3\ldots)$. A specific property of the quantum system (quantum mathematical pendulum) is the ability of redistribution of the energy level populations. The number N is the number of populated levels and it increases with the growth of the perturbation amplitude ΔI. The possibility of such a consideration at first was shown in Ref. [14].

Quantum chaos is also observed in the case of a harmonic oscillator subjected to the action of a monochromatic wave. It was shown in Ref. [15] that in the case of quantum chaos the distribution of populations by energy levels is localized and has a narrow Gaussian form [16]. In our case, we can also say that chaos is localized at a small number of levels which is defined by the perturbation amplitude.

An experimental demonstration of the above-described situation of population creeping is rather difficult because of the multiple action of detectors at moments when the system happens to be in degenerate states. As is known, each action of a detector on the system entails (at least partial) suppression of the natural course of the system development. It is obvious that the creation of a detector able to discriminate between degenerate states is also connected with a difficulty. That is why below we will try to show that it is possible to obtain a picture of population creeping without using a detector.

B. Irreversible phenomena produced by a big phase "incursion" of the probability amplitude

Different from the area of nondegenerate states G [see Eq. (96)], in the area of degenerate states G_- and G_+, the nondiagonal matrix elements of perturbation $\hat{H}'(t)$ [Eq. (97)] are not zero:

$$H'_{+-}=H'_{-+}=\langle\psi_+|\hat{H}'(t)|\psi_-\rangle\sim\int_o^{2\pi}\psi_+\psi_-^*\cos 2\varphi d\varphi\neq 0, \tag{114}$$

where the wave functions $\psi_\pm$ have been defined previously by Eqs. (37) and (38). Here, for the brevity of the notation, we omit the upper indices indicating the quantum state. An explicit dependence of $\hat{H}'(t)$ on time given by the multiplier $\cos\nu t$ is assumed to be slower as compared with the period of passages from one degenerate state to another produced by the nondiagonal matrix elements H'_{+-}. Therefore below, perturbations $\hat{H}'(t)$ will be treated as time-independent perturbations able to produce the above-mentioned passages.

Therefore in the area of degenerate states the system can be found in the time-dependent superposition state [10,13]:

$$\psi(t)=C_+(t)\psi_+ + C_-(t)\psi_-. \tag{115}$$

Probability amplitudes $C_\pm(t)$ are found by means of the following fundamental quantum-mechanical equation expressing the causality principle:

$$\begin{cases} -i\hbar\dfrac{\partial C_+}{\partial t}=(E_o+H'_{++})C_+ + H'_{+-}C_-, \\ -i\hbar\dfrac{\partial C_-}{\partial t}=H'_{+-}C_+ + (E_o+H'_{--})C_-. \end{cases} \tag{116}$$

By analogy with Eq. (45) and (46) in the case of our problem it should be assumed that $H'_{++}=H'_{--}$ and $H'_{+-}=H'_{-+}$. Let us investigate changes occurring in the state during the time ΔT while the system is in the area G_- (i.e., during the

time of movement to the left from l_-^n and, reversal, to the right to l_-^n). It will be assumed that ΔT is part of the period of modulation T.

For arbitrary initial values the system of equations (116) has a solution

$$C_+(t) = \frac{C_+(0) + C_-(0)}{2}\exp\left[\frac{-i}{\hbar}(E - H')t\right] + \frac{C_+(0) - C_-(0)}{2}\exp\left[\frac{i}{\hbar}(E - H')t\right],$$

$$C_-(t) = \frac{C_+(0) + C_-(0)}{2}\exp\left[\frac{-i}{\hbar}(E - H')t\right] - \frac{C_+(0) - C_-(0)}{2}\exp\left[\frac{i}{\hbar}(E - H')t\right], \quad (117)$$

where $E \to E_o + H'_{++}$, $E \to E_o + H'_{--}$, $H'_{+-} \to H'$.

After complementing H' in Eq. (117) with the factor $\cos \nu t$, we can take into consideration also a slow time-dependent change of perturbation ($H' \to H' \cos \nu t$).

Let the movement begin from the state ψ_- of a nondegenerate area in the close vicinity of a branching point. Then we should take

$$C_-(0) = 1,\ C_+(0) = 0 \quad (118)$$

as initial conditions.

Substituting Eq. (118) into Eq. (117), for the amplitudes $C_\pm(t)$ we obtain

$$C_-(t) = e^{(i/\hbar)Et}\cos\left(\frac{H'}{\hbar}t\right),$$

$$C_+(t) = -ie^{(i/\hbar)Et}\sin\left(\frac{H'}{\hbar}t\right). \quad (119)$$

Now, using Eq. (119), for the distribution $|\psi(t)|^2$ [Eq. (115)] we obtain

$$|\psi(t)|^2 = \cos^2\left(\frac{H'}{\hbar}t\right)|\psi_-|^2 + \sin^2\left(\frac{H'}{\hbar}t\right)|\psi_+|^2 - \frac{1}{2}\sin\left(2\frac{H'}{\hbar}t\right) \times [\psi_+(t)\psi_-^*(t) - \psi_-(t)\psi_+^*(t)]. \quad (120)$$

In the expression for $|\psi(t)|^2$ the first two terms correspond to the transition probabilities $- \to +$ and $+ \to -$, respectively, while the third term corresponds to the interference of these states. Distribution (120) corresponds to a pure state.

Note that (like any other parameter of the problem) the value H' contains a certain small error $\delta H' \ll H'$, which during the time of one passage $2\pi\hbar/H'$ leads to an insignificant correction of the phase $2\pi(\delta H'/H')$. However, during the time ΔT a phase incursion takes place and a small error $\delta H'$ may lead to uncertainty of phase $\sim(\delta H'/\hbar)\Delta T$ which may turn out to be of order 2π. In that case the phase becomes random. Therefore by the moment ΔT the distribution takes the form that can be obtained from Eq. (120) by means of averaging with respect to the random phase $\alpha = (\delta H'/\hbar)\Delta T$. Hence, after averaging expression (120), equating the interference term to zero and taking into account that $\overline{\sin^2[(H'/\hbar)t]} = \overline{\cos^2[(H'/\hbar)t]} = 1/2$ we get

$$|\psi|^2 = \overline{|\psi(t)|^2} = \tfrac{1}{2}(|\psi_+|^2 + |\psi_-|^2), \quad (121)$$

where the stroke above denotes the averaging with respect to time. The obtained formula (121) is the distribution of a mixed state, which contains probabilities of degenerate states $|\psi_\pm|^2$ with the same weights $1/2$. The assumption that a large phase is a random value that, after averaging, makes the interference term equal to zero is frequently used in analogous situations [13].

Thus we conclude that if the system remains in the areas $G_\pm$ of degenerate states for a long time, $\Delta T \gg 2\pi\hbar/H'$, $\Delta T \approx 2\pi\hbar/\delta H'$, during which the system manages to perform a great number of passages, then in the case of a passage to the nondegenerate area G the choice of continuation of the path becomes ambiguous. In other words, having reached the branch point, the system may with the same probability continue the path along two possible branches of the Mathieu characteristics. The error $\delta H'$ is evidently connected with the error of the modulation amplitude value. It obviously follows that, when passing the branch point, the mixed state (121) will transform with a $1/2$ probability to the states $ce(\varphi)$ and $se(\varphi)$, as shown in formulas (101) and (102). Analogously, we can prove the validity of all subsequent formulas for the passage probabilities (103)–(112).

VII. CONCLUSIONS

The quantum-mechanical investigation of the universal Hamiltonian (mathematical pendulum), which is reduced to the investigation of the Mathieu-Schrodinger equation, showed that on the plane (E,l) there exist three areas G_+, G_-, and G (see Fig. 2 and 4) differing from each other in their quantum properties. Motion in the area of degenerate states G_- is a quantum analog of rotatory motion of the pendulum, while motion in the area of degenerate states G_+ is an analog of oscillatory motion of the pendulum. The area G lying between G_- and G_+ can be regarded as a quantum analog of the classical separatrix. The main quantum peculiarity of the universal Hamiltonian is the appearance of branching and merging points along energy term lines. Branching and merging points define the boundaries between the degenerate areas $G_\pm$ and the nondegenerate area G. If the system defined by the universal Hamiltonian is perturbed by a slowly changing periodic field, then on the plane (E,l) the influence of this field produces the motion of the system along the Mathieu characteristics. If, moreover, the system is in degenerate areas for a sufficiently long time, then the phase incursion of wave function phases occurs while the system passes through branching points, which leads to the transition from the pure state to the mixed one. As a result of a multiple passage through branching points, the populations creep by energy terms (Fig. 5). The thus obtained mixed state can be regarded as a quantum analog of the classical stochastic layer. The number of levels affected by the irreversible creeping process is defined by the amplitude of the slowly changing field.

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